Coded Linear Precoded-OFDM versus COFDM: An Asymptotic Analysis for MMSE Equalizers

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Coded Linear Precoded-OFDM versus COFDM: an asymptotic analysis for MMSE equalizers

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Abstract

In this paper, analytic expressions of upper bounds of the bit error probability provided by the MMSE receiver are derived for OFDM based systems when the number of carriers is large enough. Considering convolutive Forward Error Correcting schemes and QPSK constellations, it is shown that coded Linear Precoded OFDM (LP-OFDM) outperforms Coded OFDM (COFDM) if the input SNR is greater than a threshold **independent of the code structure**. A confidence interval for this threshold is derived. This theoretical result is believed to be quite useful since it provides tools for choosing between COFDM and coded LP-OFDM for a given SNR.

Keywords

COFDM, MMSE equalization, MC-CDMA, LP-OFDM

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I. INTRODUCTION

When the channel is not known at the transmitter, the well-known water pouring strategy is useless in OFDM. This has motivated the use of diversity techniques and has drawn lately a lot of research interest for linear precoders (see [2], [6], [7]). Already in 1996, Wornell [8] introduced precoding as an efficient technique for transforming an arbitrary fading channel into a non-fading white Gaussian noise channel at no additional power or bandwidth expansion. He pointed out that the effects of fading could entirely be controlled by the precoder and therefore only additive noise remained for coding to control, which can in turn be achieved with comparatively shorter codes. However, Kaiser et al.[3] and Lindner et al.[4] showed *by simulations* that the diversity improvement induced by the precoding technique is considerably reduced when coding is applied with respect to COFDM. Indeed, the high redundancy of low code rates (which already performs a kind of spreading) in conjunction with the suboptimality of the MMSE receiver reduces the performance with respect to COFDM. As a matter a fact, coded LP-OFDM outperforms COFDM only for high code rates ($R > \frac{1}{2}$) and especially in the uncoded case. The first theoretical analysis of this observation was provided by Fettweis [1] based on a cut-off rate approach: he showed that coded LP-OFDM outperforms COFDM if the signal to noise ratio is greater than a threshold which only depends on the constellation size. In this contribution, we extend and confirm the analysis to the convolutive coding case.

II. LP-OFDM

A. Model

The baseband frequency domain block equivalent model of an LP-OFDM system is depicted in figure.1. The receiver front-end is formed by a symbol-matched filter followed by sampling at the symbol rate. The input symbol stream is serial to parallel converted, then the resulting N-dimensional symbol vector $\mathbf{s} = (s_1, \cdots, s_N)^T$ (a white vector process with $E\left(\mathbf{s}\mathbf{s}^H\right) = \mathbf{I}_N$) is multiplied by a $N \times N$ matrix $\mathbf{W}_{N,N}$ (in this contribution, unitary matrices with constant modulus entries such as FFT, Walsh-Hadamard are considered). This N-dimensional vector $\mathbf{x} = \mathbf{W}_{N,N}\mathbf{s}$ is parallel to serial converted, and the corresponding generated data stream is sent across a non selective Rayleigh fading channel. After serial to parallel conversion, due to intercarrier interference generated by the precoder, the N–dimensional received vector $\mathbf{y} = (y_1, \cdots, y_N)^T$ can be expressed as a function of the emitted symbol vector $\mathbf{s}$:

$$\mathbf{y} = \mathbf{H}_N \mathbf{W}_{N,N} \mathbf{s} + \mathbf{n} \tag{1}$$

where $\mathbf{n}$ is an additive white Gaussian noise such that $E\left(\mathbf{n}\mathbf{n}^H\right) = \sigma^2 \mathbf{I}_N$ and where $\mathbf{H}_N = \mathrm{diag}([\mathrm{h}_1, \ldots, \mathrm{h}_\mathrm{N}])$ is the $N \times N$ diagonal complex matrix bearing on its diagonal the channel gains. The role of matrix $\mathbf{W}_{N,N}$ is to introduce diversity so that it allows to transmit each component of $\mathbf{s}$ over N subcarriers. In the remainder of this paper, ergodic channels with Rayleigh fading are assumed. More precisely, we assume that

- $(h_k)_{k=1,\ldots,N}$ are centered unit variance identically distributed complex Gaussian random variables
- If ϕ is any function, the average $\frac{1}{N}\sum_{k=1}^{N} \phi(|\, h_k|^2)$ converges almost surely towards the common mathematical expectation $\mathrm{E}_{|\mathrm{h}|^2}(\phi(|\,\mathrm{h}|^2) = \int_0^\infty \phi(\mathrm{t})\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ of random variables $(\phi(|\, h_k|^2))_{k=1,\ldots,N}$.

The last assumption is possible in fast fading environments if time and frequency interleaving is performed on the components of $\mathbf{x} = \mathbf{W}_{N,N}\mathbf{s}$. As for the first assumption, we remark that the approach used in this paper can be immediately adapted to a non uniform power profil. Channel knowledge and perfect synchronization at the receiver is also assumed.

B. MMSE receiver

As $\mathbf{W}_{N,N}\mathbf{W}_{N,N}^H = \mathbf{I}_N$, the output of the MMSE filter is the vector $\hat{\mathbf{s}} = \mathbf{W}_{N,N}^H \mathrm{diag}\left(\frac{h_1^*}{|h_1|^2+\sigma^2}, \ldots, \frac{h_N^*}{|h_N|^2+\sigma^2}\right)\mathbf{y}$. Such a detection strategy is motivated by its very simple implementation (scalar channel equalization followed by a matrix multiplication). The k^{th} output has the following expression: $\hat{s}_k = \eta_{\mathbf{w}_k} s_k + \tau_k$ with $\eta_{\mathbf{w}_k} = \sum_{i=1}^{N} \frac{|h_i|^2 \mid w_{i,k} \mid^2}{|h_i|^2+\sigma^2}$ ($\mid w_{i,k} \mid = \frac{1}{\sqrt{N}}$ is the (i,k)th term of the unitary matrix) and τ_k is an additive noise of variance $\eta_{\mathbf{w}_k}(1-\eta_{\mathbf{w}_k})$. It has been shown in [9] that the additive noise τ_k can be considered gaussian when $N \to \infty$. Using the ergodicity assumptions formulated on the channel, in this regime, $\eta_{\mathbf{w}_k}$ has the following expression in the case of Rayleigh fading:

$$\eta = \lim_{N\to\infty} \eta_{\mathbf{w}_k} = \mathbb{E}_{|h|}\left(\frac{\mid h \mid^2}{\mid h \mid^2 + \sigma^2}\right) = 1 - \sigma^2 e^{\sigma^2} E_i(\sigma^2)$$

where $E_i(x) = \int_x^\infty \frac{e^{-u}}{u} du$ denotes the so-called exponential integral function.

III. Convolutional Coding with QPSK constellation

We study in this section the performance of the Viterbi algorithm in the case of soft decision decoding according to the method used in [5] when convolutional coding is used at the transmitter.

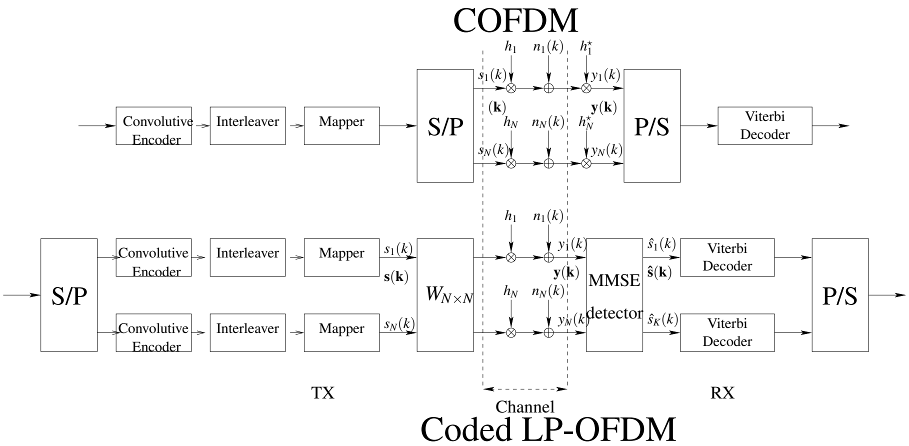


Fig. 1. COFDM an Coded LP-OFDM scheme.

A. COFDM

In classical standardized COFDM systems (see fig.1), the incoming input bit stream is first convolutionally encoded with a code rate of R, interleaved and punctured. The resulting bits are then mapped onto a QPSK constellation for forming symbols that are distributed over all the carriers. It is assumed that a soft-output Viterbi algorithm is used to decode the transmitted bits. In order to specify on which signal the Viterbi algorithm operates, we denote $(y(m))_{m\in\mathbb{Z}}$ the sequence obtained after parallel to serial conversion of the received sequence, i.e. the scalar sequence $\ldots y_1(m), y_2(m), \ldots, y_N(m), y_1(m+1), y_2(m+1), \ldots, y_N(m+1), \ldots$. Same notation is applied to variables h, s and n so that we obtain: $y(m) = h(m)s(m) + n(m)$. In COFDM systems, equalization is performed by applying the real and imaginary part operators to sequence $(\sqrt{2}\frac{h(m)^*}{|h(m)|}y(m))_{m\in\mathbb{Z}}$, the corresponding real sequence being finally de-interleaved. Thanks to the de-interleaver, the resulting real valued discrete-time signal is denoted by $z(m) = \alpha(m)b(m) + u(m)$ where α represents an i.i.d sequence of Rayleigh distributed random variables of second order moment 1, b is the BPSK sequence obtained by mapping to +1 and -1 the bits generated by the convolutional encoder, and u is an i.i.d. sequence of real Gaussian random variables of variance σ^2. The Viterbi decoding algorithm processes sequence z. In order to evaluate upper bounds on the BER, one usually first evaluates the probability $P_{COFDM}(d)$ of deciding P_1 instead of P_0 where P_0 is the path of the Viterbi algorithm trellis associated to the transmitted sequence and P_1 is a path which differs by d bits from P_0. The probability $\overline{P_{\text{COFDM}}}(d)$ has already been derived in [5] and yields:

$$\overline{P_{\text{COFDM}}}(d) \quad = \quad \frac{1}{2}[1-\sqrt{\frac{1}{1+2\sigma^2}}]-\sum_{l=1}^{d-1} C_{2l-1}^{l}\sqrt{\frac{1}{1+2\sigma^2}}\frac{1}{(2(\frac{1}{\sigma^2}+2))^l}.$$

The evaluation of the probability of the above error events leads to the following BER upper bound:

$$\boxed{\overline{P_{\text{COFDM}}} \leq \sum_{d=d_{min}}^{\infty} \frac{\beta_d}{\mu}\overline{P_{\text{COFDM}}}(d)} \tag{2}$$

Here, d_{min} is the minimal distance of the code, μ is the number of input bits in the encoder and β_d is the number of incorrectly decoded information bits, for each possible incorrect path differing from the correct one by d bits.

B. Coded LP-OFDM

In the context of coded LP-OFDM (see fig.1), one Viterbi decoder is applied on each subband k and processes the real and imaginary parts of the signal at the output of the Wiener filter. Indeed, after MMSE equalization, the noise is correlated across the carriers: taking into account the matrix correlation in the viterbi algorithm would increase exponentially the number of states. Therefore, each substream is encoded separately. Denote $z_k(m)$ the corresponding real symbol. According to section II, $z_k(m) = \eta b_k(m) + u_k(m)$, b_k is the BPSK sequence obtained by mapping the bits at the output of the convolutional encoder, and u_k is a real Gaussian i.i.d. of variance ${\sigma^2}_{\text{both}} = \eta(1-\eta)$. Since η is deterministic, the bit error probability in the pairwise comparison of two paths that differ by d bits is given by :

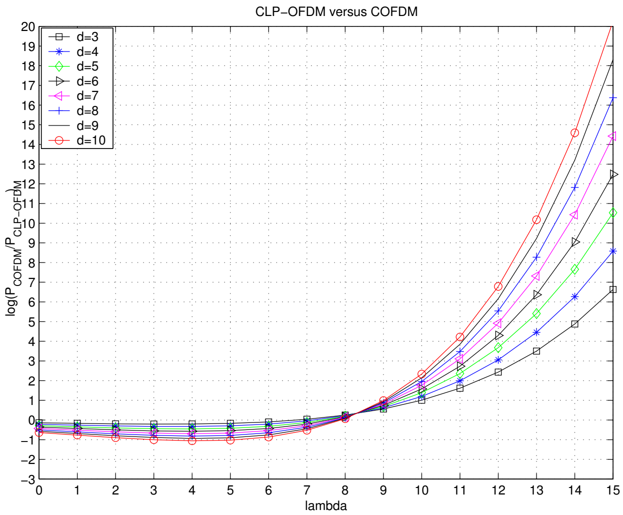
$$\overline{P_{\text{CLP-OFDM}}}(d) \quad = \quad P\left(N(0,1) > \frac{\sum_{l=1}^{d}\eta}{\sqrt{\sum_{l=1}^{d}\eta(1-\eta)}}\right) = Q\left(\sqrt{\frac{d\eta}{1-\eta}}\right)$$

$N(0,1)$ is zero mean Gaussian variable with unit variance. The overall error probability is thus bounded by:

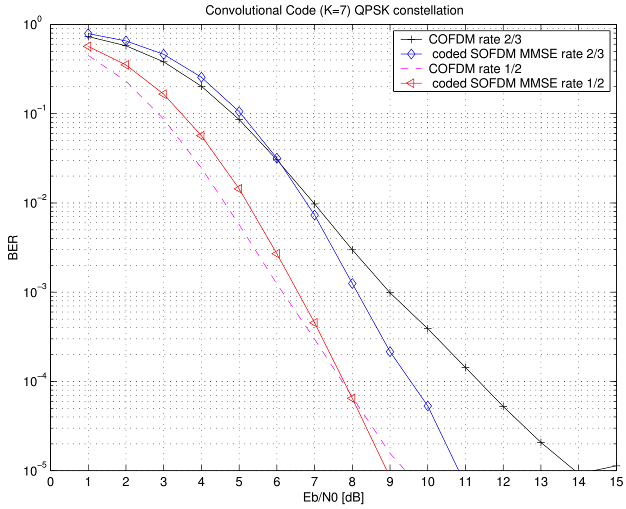
$$\boxed{\overline{P_{\text{CLP-OFM}}} \leq \sum_{d=dmin}^{\infty} \frac{\beta_d}{\mu}\overline{P_{\text{CLP-OFDM}}}(d)} \tag{3}$$

C. COFDM versus coded LP-OFDM

In the previous section, we have derived bounds to evaluate the bit-error probability knowing the transfer function of the code (i.e β_d and μ) . However, the most important point of this section lies in the observation that the difference $l(d) = \overline{P_{\text{COFDM}}}(d) - \overline{P_{\text{CLP-OFDM}}}(d)$ is positive for all $d > 3$ if the input SNR λ is greater than a threshold. This is an important issue since formulas (2) and (3) in conjunction with the above properties of $l(d)$ imply that coded LP-OFDM outperforms COFDM if the SNR is higher than a threshold independent of the particular chosen code. This claim is illustrated in the following figures.



$\log(P_{COFDM}/P_{CLP-OFDM})$ versus λ.



Coded LP-OFDM-COFDM.

In the first figure, we have plotted $\log(P_{\text{COFDM}}(d)/P_{\text{CLP-OFDM}}(d))$ versus the input SNR $\lambda = \frac{1}{\sigma^2}$. We observe that for a fixed distance d, there exists a threshold $\lambda_{lim}(d)$ such as $l(d) < 0$ for $\lambda < \lambda_{lim}(d)$ while $l(d) > 0$ for $\lambda > \lambda_{lim}(d)$. In fact, the threshold $\lambda_{lim}(d)$ (corresponding to the solution of equation $l(d) = 0$) varies in a tight interval ($7.5dB < \lambda_{lim}(d) < 8.1dB$ for all $d > 3$). In practice, these results show that regardless of the code structure (i.e the expression of β_d in our previous analysis), coded LP-OFDM outperforms COFDM if the signal to noise ratio is greater than a threshold included in a tight interval. These results gives us tools to choose between coded LP-OFDM and COFDM for a given couple $(R, \frac{Eb}{No})$ and confirms that a trade-off between spreading and coding is to be considered when designing a coded LP-OFDM scheme. The relation between the input signal to noise ratio λ_{lim}, $\left(\frac{Eb}{No}\right)_{lim}$ and the code rate R is : $\lambda_{lim} = 2 * R * \left(\frac{Eb}{No}\right)_{lim}$ (the number of bits conveyed by a QPSK modulation is 2). For a fixed threshold λ, as R decreases, the threshold $\frac{Eb}{No}$ increases. The high redundancy of low code rates introduces a kind of information spreading in COFDM. This favors COFDM with respect to coded LP-OFDM as characterized by the higher threshold in terms of $\frac{Eb}{No}$. Our analysis thus confirms the results observed by Kaiser [3] and the theoretical discussion on the cut-off rate made by Fettweis [1] . The second figure plots the simulated performance of respectively two constraint length $K = 7$ convolutional encoders rate $\frac{1}{2}$ and $\frac{2}{3}$. For each code, a threshold is observed which belongs to the confidence interval.

- For $R = \frac{2}{3}$, the threshold is: 6.5dB (Confidence interval 6.2dB-6.8dB)
- For $R = \frac{1}{2}$, the threshold is: 8dB (Confidence interval 7.5dB-8.1dB)

IV. Conclusion

In this paper, we have analyzed the performance of convolutionnaly coded LP-OFDM and shown that the latter outperforms COFDM if the signal to noise ratio is greater than a given threshold which has been determined analytically.

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